Stochastic models in a problem of the Caspian sea level forecasting

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ABSTRACT

In hydrological applications, the problem of the definition of a type of the stochastic model of the process under investigation and its parameters estimation is important. One of the most interesting cases is the closed water body level forecasting. Being the integrated characteristic, the level of a closed water body is rather sensitive to the behavior of the processes determining the inflow and the outflow components of the water balance on long time intervals.

To solve the problem of forecasting of the Caspian Sea level fluctuations, both Langevin approach to the solution of the stochastic water balance equation and the diffusion theory of Fokker-Planck- Kolmogorov are used.

For the description of river runoff fluctuations there are used:
- the solution of Markov equation in the form of the bilinear decomposition on systems of orthogonal functions;
- stochastic differential equations (SDE) in the form Ito or Stratonovich;
- diffusion equations of Fokker-Planck-Kolmogorov;

1. INTRODUCTION

In hydrological applications, the problem of the definition of a type of the stochastic model of the process under investigation and its parameters estimation is important. One of the most interesting cases is the closed water body level forecasting. Being the integrated characteristic, the level of a closed water body is rather sensitive to the behavior of the processes determining the inflow and the outflow components of the water balance on long time intervals.

The researches carried out during recent decades have shown the description of the runoff fluctuations as the simple Markov chain to be acceptable.. To solve the problem of forecasting of the Caspian Sea level fluctuations, both Langevin approach to the solution of the stochastic water balance equation and the diffusion theory of Fokker-Planck- Kolmogorov are used.

For the description of river runoff fluctuations there are used:
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- stochastic differential equations (SDE) in the form Ito or Stratonovich;
- diffusion equations of Fokker-Planck- Kolmogorov;

Let us consider the listed problems in more detail.

2. THE SOLUTION OF MARKOV EQUATION IN THE FORM OF BILINEAR DECOMPOSITION ON SYSTEMS OF ORTHOGONAL POLYNOMS.

The two-dimensional density satisfies to Markov equation (under some conditions) if it the sum of the following kind [3]:

$$p(t,x,y) = p(x)p(y)\left[1+\sum_{k=1}^{\infty} e^{-\lambda_k t}\varphi_k(x)\varphi_k(y)\right] \quad (1)$$

where λ_k are positive numbers, so as $0 < \lambda_1 \le \lambda_2 \le \lambda ... \le \lambda_k < ...$,

and $\varphi_k(x)$ and $\varphi_k(y)$ form the system of orthogonal functions with the weight $p(x)$

Generalization of (1) on the case of the so-called two-parametric gamma distribution has been developed by E.S.Blohinov and O.V.Sarmanov. The two-dimensional density for

$p(x) = \frac{\gamma^{\gamma}}{\tilde{A}(\gamma)} x^{\gamma-1} e^{-\gamma x}$ will be written as [1]:

$$f(x,y)= p(x)p(y)\left[1+\sum_{1=1}^{n} R^k L_k^{\gamma-1}(\gamma x) L_k^{\gamma-1}(\gamma y)\right], \quad (2)$$

where $L_n^{\alpha} = \tilde{L}_n^{\alpha} \cdot \sqrt{\dfrac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}}$.

3. DIFFUSION PROCESSES.

The transition probability density f that satisfies to Markov equation also satisfies to the inverse Fokker-Planck-Kolmogorov (FPK) equation, which looks like:

$$\frac{\partial f}{\partial s} = -a(s,x)\frac{\partial f}{\partial x} - \frac{b\ (s,x)}{2}\frac{\partial^2 f}{\partial x^2}, \quad (3)$$

Where $a\ (s, x)$ is called a drift coefficient, and $b\ (s, x)$ - a diffusion coefficient.

The two-dimensional density that represents the solution of the equation (3) generates the diffusion Markov process.

THE STOCHASTIC DIFFERENTIAL EQUATIONS.

Following [4], let us consider the case when some system is described by the following differential equation:

$$\frac{dh}{dt} = f(h,t) + g(h,t)\cdot n(t) \quad , \quad (4)$$

with the given initial conditions: $h(t_0) = h_0$, $n(t)$ is normal white noise.

If $h(t)$ is the diffusion Markov process with drift coefficients $a\ (h,\ t)$ and diffusion coefficients $b\ (h,\ t)$, then for the stochastic differential equation (SDE) the following relations are true:

$$a(h,t) = f(h,t) + \frac{N_0}{4} g(h,t)\frac{\partial g(h,t)}{\partial h} \quad (5)$$

$$b(h,t) = N_0\, g^2(h,t)/2 \quad (6)$$

Here N_0 is the white noise intensity.

The solution of SDE, received in such a way, can be obtained by two methods. The first method named the Langevin one assumes the notation of SDE solution by quadratures and the consideration of this solution as some operator, transforming the input random process into the output one.

The other idea will consist in an identification of the written equation of the system with the stochastic differential equation and in the calculation of the FPK equation coefficients with its subsequent solution.

4. MARKOV PROCESS WITH TWO-PARAMETRICAL A PRIORI GAMMA-DISTRIBUTION.

Having got two-dimensional density, let us receive corresponding coefficients of the FPK equation and then SDE.

So, for the one-dimensional distribution law

$$p(x) = \frac{\gamma^{\gamma}}{\tilde{A}(\gamma)} x^{\gamma-1} e^{-\gamma x} \quad (7)$$

Let us calculate the drift coefficient in the FPK equation:

$$a(x) = -\mu(x-1) \quad (8)$$

$$b(x) = \frac{2\mu x}{\gamma} \quad (9)$$

Let us write the stochastic differential equation as

$$\frac{dx}{dt} = -\mu(x-1) - \frac{\mu}{2\gamma} + 2\sqrt{\frac{\mu x}{\gamma N_0}}\, n(t) \quad (10)$$

5. MODELLING OF PSEUDO-RANDOM VARIABLES ACCORDING TO THE SCHEME OF THE MARKOV GAMMA-PROCESS.

Let us further consider the third approach based on the difference approximation of the solution of the stochastic differential equation [4]:

$$h_{i+1} = h_i + f_i\Delta t + g_i\Delta v_i + \frac{1}{2} g_i\left(\frac{\partial g}{\partial h}\right)_i \Delta v_i^2 +$$
$$\frac{1}{2}\left[g_i\left(\frac{\partial f}{\partial h}\right)_i + f\left(\frac{\partial g}{\partial h}\right)_i + \left(\frac{\partial g}{\partial t}\right)_i\right]\Delta t\Delta v_i + \quad (11)$$
$$+\frac{1}{u}\left[g_i\left(\frac{\partial g}{\partial h}\right)_i^2 + \frac{1}{2} g_i^2\left(\frac{\partial^2 g}{\partial h^2}\right)_i\right]\Delta v_i^3$$

For calculations using (11) it is necessary to calculate derivatives on each time step and to simulate the sequence of Wiener process increments with the given parameters.

6. THE SOLUTION OF THE STOCHASTIC DIFFERENTIAL EQUATION OF THE CASPIAN SEA WATER BALANCE.

The differential equation of water balance of Caspian Sea looks like [2]:

$$\frac{dh}{dt} = -\alpha h(t) + g(t) \quad , \quad (12)$$

where h -is the sea level; α - is the parameter dependent on the coast steepness and g is the resulting of the processes of inflow and evaporation from sea surface minus the precipitation on its surface. It is supposed also, that g is the stationary Markov process with autocorrelation coefficient and a dispersion known.

The solution of the equation (12) is possible to be found in two ways. One of them is the already mentioned Langevin approach considered in [2] in detail.

At realization of Langevin approach it is difficult to receive a form of conditional distribution because the corresponding equations are written for parameters (moments) of distribution.

For the numerical solution of the equation (12) let us use its difference approximation and the corresponding algorithm considered above.

Using random numbers generation algorithm, let us receive realizations of the sea level course for 50 years forward with the same initial condition. The results of such numerical experiment are presented in table 1. The comparison of the results of two approaches applied to the solution of the probability forecasting problem of the Caspian sea level shows, that the limitation by Markov approximation results in regular overestimate of conditional dispersion for the forecasting period about 10 years and less. At big time period the results practically coincide, that as a whole corresponds to the assumptions made earlier.

Table 1. Results of numerical solution of CDE of the Caspian Sea balance (38) with initial condition $H_0 = 1$ m.

Time period, years	Conditional average	Conditional mean square deviation (MSD)	Asymmetry	Conditional MSD for Langevin approach
1	0.77	0.21	0.02	0.13
5	0.69	0.43	0.06	0.39
10	0.58	0.57	0.02	0.54
20	0.45	0.71	-0.01	0.69
30	0.32	0.74	-0.01	0.76
40	0.23	0.79	-0.04	0.79
50	0.16	0.80	0.07	0.81

7. CONCLUSIONS.

- For modelling the runoff and evaporation processes, the stochastic differential equation is offered generating the so-called Markov gamma-process with the linear regression equation.
- The description of the Caspian sea level dynamics in frameworks of the diffusion approximation (Fokker-Planck-Kolmogorov equation) is acceptable for forecast time of more than 10 years.

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